

# Simplicial minisuperspace models in the presence of a massive scalar field with arbitrary scalar coupling $\eta R\phi^2$

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## ABSTRACT

We consider a simplicial minisuperspace model based on a cone over the  $\alpha_4$  triangulation of the 3–sphere, in the presence of a massive scalar field,  $\phi$ , with arbitrary scalar coupling term  $\eta R\phi^2$ . By restricting all the interior edge lengths and all the boundary edge lengths to be equivalent and the scalar field to be homogenous on the 3–space, we obtain a two dimensional minisuperspace model  $\{s_i, \phi_i\}$  for what is one of the most relevant triangulations of the spatial universe. We solve the two classical equations and find that there are both real Euclidean and Lorentzian classical solutions for any size of the boundary 3–space,  $\alpha_4$ . After studying the analytic properties of the action in the space of complex edge lengths we then obtain steepest descents contours of constant imaginary action passing through Lorentzian classical geometries giving the dominant contribution and yielding a convergent wavefunction of the universe. We also show that the semiclassical wavefunctions for the Euclidean solutions associated with large boundary 3–spaces are exponentially suppressed,

Consequently we can be confident that by using the SD contour associated with classical Lorentzian solutions the semiclassical approximation based on those classical solutions is well justified, clearly predicting classical spacetime in the late universe. This wavefunction is then evaluated numerically.

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# 1 Introduction

The sum over histories formulation of quantum gravity has undoubtedly been one of the most useful tools in the long running program of quantising gravity. In this formulation an amplitude for a certain state of the universe is constructed by summing over a certain class of physically distinct histories that satisfy appropriate boundary conditions, weighted by their respective action. There are no a priori definitions of what the class of histories and boundary condition should be. As such what we have are proposals for both of them. In our opinion Hartle-Hawking's no boundary proposal formulated on [1] is the most natural one. As for the space of histories, until recently it was widely thought that these histories should be confined to smooth manifolds with well-behaved metrics. However, in [2] and [3], Schleich and Witt put forward a powerful case for the generalisation of the space of histories to include smooth conifolds. Computations for concrete models in [4], [5] and by us [6] have reinforced that proposal.

However, even this generalisation of the Hartle-Hawking proposal is plagued by at least two main problems. Firstly, the Euclidean gravitational action is not bounded from below which leads to the divergence of the Euclidean path integral. Secondly, there is no clear prescription for the correct integration contour to use. In [7], Hartle proposed the use of the steepest descents contour in the space of complex metrics as the solution to both problems. Furthermore by choosing the steepest descents contour passing through the classical solutions of the theory, he made it very likely that the path integral be dominated by classical four-geometries, i.e., solutions of Einstein's equations and stationary points of the path integral, as desired for any wave function that is intended to represent our current Universe. Note that in this view, the fact that an integration solely over real-valued Euclidean geometries does not yield a convergent result for the path integral, is actually a good thing, for such a path integral would never predict the oscillatory behaviour in the late Universe that traditionally represents classical Lorentzian space-time.

However, the usefulness of the above formulation in computational terms leaves something to be desired. The resulting functional integrations over the metric tensor and matter fields are usually very hard, no matter what integration contour we choose. It is here that a simplicial formulation can be of great help. As we shall see below, in less than seven dimensions smooth manifolds are in a one-to-one correspondence with a special kind of simplicial complexes called combinatorial manifolds. This means that we can substitute a sum

over smooth manifolds and metric tensors by a sum over simplicial complexes and their squared edge lengths, which in certain cases greatly simplifies calculations. However, as in the continuum case, a complete sum is still very hard, so we also end up using approximate minisuperspace models with features that greatly reduce the complexity of the calculations, in particular simplicial minisuperspace models based in Regge calculus. Such simplicial minisuperspace models were introduced by Hartle [8]. One typically takes the simplicial complex which models the topology of interest to be fixed and the square edge length assignments play the role of the metric degrees of freedom. The summation over edge lengths models the continuum integration over the metric tensor. This approach has several advantages. First by treating the four-geometry directly it is more adequate to deal with the Hartle-Hawking proposal, [1], (with its four-dimensional nature), than the usual  $3 + 1$  ADM decomposition of space-time, where a careful study of how the four-geometry closes off at the beginning of the universe is essential. Second, by discretizing space-time the classical equations become algebraic which makes it easier to find classical solutions which are essential to the semiclassical approximation. Third, the simplicial minisuperspace models offer the possibility of systematic improvement.

In the present case we consider one such model where the topology has been fixed to that of a cone over  $\alpha_4$ , which is the simplest triangulation of the 3-sphere. As a matter sector we consider a massive scalar field with arbitrary scalar coupling to gravity,  $\eta R\phi^2$ .

Unlike the results of our previous papers [6] and [9] where the only real classical solutions for universes with large  $3 - D$  boundary, (like our own), were Lorentzian solutions, the introduction of the scalar coupling leads to the additional existence of real Euclidean solutions for universes with boundary of arbitrarily large size. However, the semiclassical wavefunction associated with such solutions proves to be exponentially suppressed and so the Lorentzian solutions still dominate, as they should if our model is to predict classical spacetime for the late universe.

## 2 Simplicial Framework

The crucial point in implementing any sum over histories formulation of quantum gravity is the specification of the set of histories to be considered. A history in quantum gravity is specified by its topology, smooth structure and geometry. In the case of the Hartle-Hawking approach, the histories considered have been topological spaces with the topology and smooth

structure of a smooth compact manifold and a geometry specified on that manifold. Lately several papers like [5] and [6], have pointed to the advantages of extending such space of histories to include histories having more general topology, namely conifolds. In this paper we shall not deal with this issue.

The traditional choice of histories described above derives from the fact that, in the (Euclideanized) classical theory of gravity a classical history is a Riemannian manifold  $(M^n, A, g)$ , where  $g$  is a Riemannian metric and:

**Definition 2.1** *The pair  $(M^n, A)$ , where  $A = \{(U_a, \Phi_a)\}$ , is a smooth manifold with atlas  $A$  if it satisfies the following conditions:*

- *Every point of  $M^n$  has a neighbourhood  $U_a$  which is homeomorphic to an open subset of  $R^n$ , via a mapping*

$$\Phi_a : U_a \rightarrow R^n$$

- *Given any two neighbourhoods with nonempty intersection, then the mapping*

$$\Phi_b \Phi_a^{-1} : \Phi_a(U_a \cap U_b) \rightarrow \Phi_b(U_a \cap U_b)$$

*is a smooth mapping between subsets of  $R^n$ .*

A topological space that satisfies only the first condition is called a topological manifold. Such spaces are not appropriate as histories because the lack of a smooth structure, i.e., atlas, makes it impossible to define essential concepts on them, like distance, continuous and differentiable functions (like scalar fields), integration, etc.

The concrete implementation of a sum over smooth manifolds is very difficult. One of the main problems is how to provide a finite representation for the manifold-based histories. A simplicial formulation of quantum gravity aims to provide one such representation. For that to happen one must prove that there is a one to one correspondence between the set of smooth manifolds and some set of simplicial complexes.

A simplicial complex somehow plays the role of topological manifold in the simplicial framework, in the sense that it also lacks the necessary structure to define essential concepts like dimension, distance, volume, curvature etc.

**Definition 2.2** A simplicial complex  $(K, |K|)$  is a topological space  $|K|$  and a collection of simplices  $K$ , such that

- $|K|$  is a closed subset of some finite dimensional Euclidean space.
- If  $\sigma$  is a face of a simplex in  $K$ , then  $\sigma$  is also contained in  $K$ .
- If  $\sigma_a$  and  $\sigma_b$  are simplices in  $K$ , then  $\sigma_a \cap \sigma_b$  is a face of both  $\sigma_a$  and  $\sigma_b$ .
- The topological space  $|K|$  is the union of all simplices in  $K$ .

If we are to be able to define essential concepts such as continuity and differentiability of functions on simplicial complexes we will need to introduce some kind of structure similar to that of smooth manifolds. To do so a few more definitions are necessary.

Remember that the PL-join of a point  $a$  with a set  $L$ , denoted  $aL$ , is the union of all line segments joining points of  $L$  with the point  $a$ . This is also called a PL-cone over  $L$  with apex  $a$ . Remember also that:

**Definition 2.3** A map  $f : P \rightarrow Q$  between two polyhedra  $P$  and  $Q$ , is said to be piecewise linear, (PL), if each point  $p$  in  $P$  has a cone neighbourhood  $N = aL$  such that

$$f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$$

where  $x$  is in  $L$ ,  $a$  is the apex of the cone  $N$  and  $\lambda, \mu \geq 0, \lambda + \mu = 1$ .

**Definition 2.4** A PL manifold  $M^n$  is a topological manifold endowed with a PL-atlas  $A = (U_a, \Phi_a)_{a \in \Lambda}$ , such that the mapping

$$\Phi_b \Phi_a^{-1} : \Phi_a(U_a \cap U_b) \rightarrow \Phi_b(U_a \cap U_b) \quad (1)$$

is a piece-wise linear (PL) mapping between subsets of  $R_+^n$ .

These PL-manifolds are very closely connected with a special kind of simplicial complexes, the combinatorial manifolds, which are defined by imposing additional restrictions to the very general definition of simplicial complex. These restrictions make it possible to define concepts like distance, volume, curvature, etc.

**Definition 2.5** A combinatorial  $n$ -manifold  $\mathcal{M}^n$ , is an  $n$ -dimensional simplicial complex such that

- *It is pure.*
- *It is non-branching.*
- *Any two  $n$ -simplices can be connected by a sequence of  $n$ -simplices, each intersecting along some  $(n - 1)$ -simplex.*
- *The link of every vertex is a combinatorial  $(n - 1)$ -sphere.*

Indeed it can be shown that PL manifolds are equivalent to combinatorial manifolds, [3]. So every combinatorial manifold admits a PL-atlas  $(U_a, \Phi_a)_{a \in \Lambda}$ , by which concepts like continuity and differentiability of scalar fields in combinatorial manifolds can be defined, thus playing a similar role to that of the smooth structure in smooth manifolds.

Furthermore given these definitions, following [2] it can be proven that in less than seven dimensions every *PL*-manifold has a *unique* smoothing so we can state a very important result, namely:

In less than seven dimensions, every smooth manifold,  $M^n$ , is triangulated by a *unique* combinatorial manifold,  $\mathcal{M}^n$ .

Obviously each smooth manifold has several distinct triangulations, what this result says is that only one of them is a combinatorial triangulation, i.e., a triangulation based on a combinatorial manifold, and not just any simplicial complex.

So we see that the topological part of the sum over histories can be recast in terms of simplicial representatives of the “continuum” spaces. However in order to define a concrete sum over simplicial histories we still need to associate a metric and an action to each simplicial complex to be considered. Note that up to now we have not specified any kind of metric information associated with simplicial complexes. Once we have fixed the topology of the underlying simplicial complex, the most convenient way to attach metric information to it, is to use Regge calculus.

## 2.1 Regge calculus

A convenient way of defining an  $n$ -simplex is to specify the coordinates of its  $(n + 1)$  vertices,  $\sigma = [0, 1, 2, \dots, n]$ . By specifying the squared values of the lengths of the edges  $[i, j]$ ,  $s_{ij}$ , we fix the simplicial metric on the simplex:

$$g_{ij}(s_k) = \frac{s_{0i} + s_{0j} - s_{ij}}{2} \quad (2)$$

where  $i, j = 1, 2, \dots, n$ .

So if we triangulate a smooth manifold  $M$  endowed with a metric  $g_{\mu\nu}$  by a homeomorphic simplicial manifold  $\mathcal{M}$ , the metric information is transferred to the simplicial metric of that simplicial complex

$$g_{\mu\nu}(x) \longrightarrow g_{ij}(\{s_k\}) = \frac{s_{0i} + s_{0j} - s_{ij}}{2} \quad (3)$$

In the continuum framework the sum over metrics is implemented through a functional integral over the metric components  $\{g_{\mu\nu}(x)\}$ . In the simplicial framework the metric degrees of freedom are the squared edge lengths, and so the functional integral is replaced by a simple multiple integral over the values of the edge lengths. But not all edge lengths have equal standing. Only the ones associated with the interior of the simplicial complex get to be integrated over:

$$\int Dg_{\mu\nu}(x) \longrightarrow \int D\{s_i\} = \prod \int d\mu(s_i) \quad (4)$$

The boundary edge lengths remain after the sum over metrics and become the arguments of the wavefunction of the universe. In the simplicial framework the fact that the geometry of the complexes is completely fixed by the specification of the squared values of all edge lengths, means that all geometrical quantities, such as volumes and curvatures, can be expressed completely in terms of those edge lengths.

We shall also be considering a scalar field with arbitrary mass and scalar curvature coupling, taking values  $\phi_k$  at each vertex  $k$  of the complex. Generally these values will be independent, but like the edge lengths, not all have equal standing. Only the ones associated with interior vertices,  $\{\phi_i\}$  are to be integrated over:

$$\int d\phi \longrightarrow \int D\{\phi_i\} = \prod \int d\phi_i \quad (5)$$

The values of the field at the boundary vertices  $\{\phi_b\}$  are just boundary conditions, becoming the arguments of the wavefunction.

The Euclideanized Einstein action for a smooth 4-manifold  $M$  with boundary  $\partial M$ , and endowed with a 4-metric,  $g_{\mu\nu}$ , and a scalar field  $\phi$  with arbitrary mass  $m$  and scalar curvature coupling constant  $\eta$  is

$$\begin{aligned}
I[M, h_{ij}, \phi] &= - \int_M d^4x \sqrt{g} \frac{(R - 2\Lambda)}{16\pi G} - \int_{\partial M} d^3x \sqrt{h} \frac{K}{8\pi G} + \\
&+ \frac{1}{2} \int_M d^4x \sqrt{g} (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \eta R \phi^2)
\end{aligned}$$

where  $K$  is the extrinsic curvature.

Its simplicial analogue will be the Regge action for a combinatorial 4-manifold,  $\mathcal{M}$ , with squared edge lengths  $\{s_k\}$ , and with a scalar field taking values  $\{\phi_v\}$  for each vertex  $v$  of  $\mathcal{M}$ , [10]:

$$\begin{aligned}
I[\mathcal{M}, \{s_k\}, \{\phi_v\}] &= \frac{-2}{16\pi G} \sum_{\sigma_2^i} V_2(\sigma_2^i) \theta(\sigma_2^i) + \frac{2\Lambda}{16\pi G} \sum_{\sigma_4} V_4(\sigma_4) \\
&- \frac{2}{16\pi G} \sum_{\sigma_2^b} V_2(\sigma_2^b) \psi(\sigma_2^b) + \frac{1}{2} \sum_{\sigma_1=[ij]} \tilde{V}_4(\sigma_1) \frac{(\phi_i - \phi_j)^2}{s_{ij}} \\
&+ \frac{1}{2} \sum_j \tilde{V}_4(j) m^2 \phi_j^2 + \frac{1}{2} \sum_j \tilde{V}_4(j) \eta R_j \phi_j^2
\end{aligned}$$

where:

- $\sigma_k$  denotes a  $k$ -simplex belonging to the set  $\Sigma_k$  of all  $k$ -simplices in  $\mathcal{M}$ .
- $\theta(\sigma_2^i)$ , is the deficit angle associated with the interior 2-simplex  $\sigma_2^i = [ijk]$

$$\theta(\sigma_2^i) = 2\pi - \sum_{\sigma_4 \in St(\sigma_2^i)} \theta_d(\sigma_2^i, \sigma_4) \quad (6)$$

and  $\theta_d(\sigma_2^i, \sigma_4)$  is the dihedral angle between the 3-simplices  $\sigma_3 = [ijkl]$  and  $\sigma_3' = [ijkm]$ , of  $\sigma_4 = [ijklm]$  that intersect at  $\sigma_2^i$ . Its full expression is given in [4].

- $\psi(\sigma_2^b)$  is the deficit angle associated with the boundary 2-simplex  $\sigma_2^b$ :

$$\psi(\sigma_2^b) = \pi - \sum_{\sigma_4 \in St(\sigma_2^b)} \theta_d(\sigma_2^b, \sigma_4) \quad (7)$$

- $V_k(\sigma_k)$  for  $k = 2, 3, 4$  is the  $k$ -volume associated with the  $k$ -simplex,  $\sigma_k$ , and once again their explicit expressions in terms of the squared edge lengths are given in [4].



- $\tilde{V}_4(\sigma_1)$ , is the 4-volume in the simplicial complex  $\mathcal{M}$ , associated with the edge  $\sigma_1$ , i.e., the volume of the space occupied by all points of  $\mathcal{M}$  that are closer to  $\sigma_1$  than to any other edge of  $\mathcal{M}$ . The same holds for  $\tilde{V}_4(j)$  where  $j$  represents all vertices of  $\mathcal{M}$ .
- It can be shown, [10], that

$$\tilde{V}_4(j)R_j = \frac{2}{3} \sum_{\sigma_2[jkl]} V_2(\sigma_2)\chi(\sigma_2)$$

where the sum is over all triangles  $\sigma_2$  that contain the vertex  $j$ , and  $\chi(\sigma_2)$  is the deficit angle associated with  $\sigma_2$ . We use the new symbol  $\chi$  because the triangle  $\sigma_2$  can be an interior or boundary triangle.

It is easy to see that both  $\tilde{V}_4(\sigma_1)$  and  $\tilde{V}_4(j)$ , can be expressed exclusively in terms of the edge lengths  $\{s_k\}$ . In fact all the previous terms can be written exclusively in terms of  $\{s_k\}$  and  $\{\phi_k\}$ .

So we see that any history in QG of the type  $(M^4, A, g_{\mu\nu}, \phi)$ , where  $M^4$  represents a topological manifold endowed with a smooth structure  $A$ , metric  $g_{\mu\nu}$ , and in the presence of matter fields represented by  $\phi$ , has an *unique* simplicial analogue,  $(\mathcal{M}^4, \{s_k\}, \{\phi_j\})$ . This allows us to concretely implement the formal sum over histories in terms of this finite representation as:

$$\Psi[\partial\mathcal{M}, \{s_b\}, \{\phi_b\}] = \sum_{\mathcal{M}^4} \int D\{s_i\} D\{\phi_i\} e^{-I[\mathcal{M}^4, \{s_i\}, \{s_b\}, \{\phi_i\}, \{\phi_b\}]} \quad (8)$$

where

- $\{s_i\}$  are the squared lengths of the interior edges
- $\{s_b\}$  are the squared lengths of the boundary edges
- $\{\phi_i\}$  are the values of the field at the interior vertices
- $\{\phi_b\}$  are the values of the field at the boundary vertices

Although the functional integral over metrics has been written explicitly in terms of the edge lengths, this expression is still heuristic because we still need to specify the list of suitable simplicial complexes  $\mathcal{M}^4$  we intend to sum over, the measure, and the integration contour to be used. To circumvent these problems we shall compute the sum approximately

by singling out a subfamily of simplicial histories described by only a few parameters and carrying out the sum over these histories alone.

An example of this is to adopt a simplicial minisuperspace approximation. We now describe in some detail the minisuperspace model we shall consider.

### 3 Simplicial Minisuperspace

We shall reduce our attention to a significant subfamily of simplicial histories characterised by the following restrictions:

We shall consider that the universe has only one  $S^3$  boundary and it is well approximated as a simplicial cone over the closed combinatorial 3-manifold  $\alpha_4$ , which is the simplest triangulation of the 3-sphere,  $S^3$ .

$$\mathcal{M}^4 = a * \alpha_4 \tag{9}$$

The combinatorial manifold  $\alpha_4$  has been described in detail elsewhere [8]. We can see a representation of it in figure 1. It has 5 vertices, each connected to all others.

Note that since all vertices of  $\mathcal{M}^4$ , (even the interior one) have combinatorial links that are homeomorphic to a 3-sphere,  $\mathcal{M}^4$  is a combinatorial 4-manifold.

- By using a cone-like structure, translated by the existence of only one interior vertex, the apex  $a$  which we shall henceforth denote as 0, the only boundary of  $\mathcal{M}^4$  is  $\alpha_4$ . So it is very easy to define boundary simplices and interior simplices. If a simplex contains the interior vertex 0 it is an interior simplex if not it is a boundary simplex. Moreover, all interior  $p$ -simplices in  $\mathcal{M}^4$  are cones over  $(p-1)$ -simplices of  $\alpha_4$  with apex 0.
- If we label the five boundary vertices of  $\mathcal{M}^4$  simply as 1, 2, 3, 4, 5, then the cone-like structure of  $\mathcal{M}^4$  leads to all interior edges being of the same form  $[0, b]$ , with  $b = 1, 2, 3, 4, 5$ . So it makes sense to introduce the restriction that all interior edges have equal lengths whose squared value is denoted  $s_i = s_{0b}$ . A similar assumption is made with respect to the boundary edge lengths, i.e., we consider them all to be equal to a common value  $s_{ij} = s_b$ , with  $i, j = 1, 2, 3, 4, 5$ . We thus obtain an *isotropic and homogeneous* triangulation of the 4-universe. This leads to an enormous simplification in the metric part of the integration for the wavefunction (8),

since the multiple integral  $\int D\{s_i\}$  is reduced to a single integral  $\int ds_i$ . It also greatly simplifies the expression of the simplicial action since there will only be one type of boundary and interior triangle.

- The simplifications assumed with respect to the edge lengths make it natural to assume that the scalar field is spatially homogeneous and isotropic. So we assume that the scalar field takes the same value  $\phi_b$  for all boundary vertices of  $\mathcal{M}^4$ . The value at the interior vertex,  $\phi_i$ , is independent. Again, this leads to an simplification in the matter fields part of the integration for the wavefunction (8), since the multiple integral  $\int D\{\phi_i\}$  is reduced to a single integral  $\int d\phi_i$ .

### 3.1 Minisuperspace Wavefunction

We can now concretely implement a simplicial minisuperspace approximation to the wavefunction of the universe of the type (8), as

$$\Psi[\alpha_4, s_b, \phi_b] = \int ds_i d\phi_i e^{-I[\alpha_4, s_i, s_b, \phi_i, \phi_b]} \quad (10)$$

The Regge action for this minisuperspace can now be calculated. For simplicity we introduce rescaled metric variables:

$$\xi = \frac{s_i}{s_b} \quad (11)$$

$$S = \frac{H^2 s_b}{l^2} \quad (12)$$

where  $H^2 = l^2 \Lambda / 3$ , and  $l^2 = 16\pi G$  is the Planck length. We shall work in units where  $c = \hbar = 1$ .

Then the volume of the 4-simplices in  $\mathcal{C}^4$  is

$$V_4(\sigma_4) = \frac{l^4}{24\sqrt{2}H^4} S^2 \sqrt{\xi - 3/8}. \quad (13)$$

The volume of the 10 internal 2-simplices,  $\sigma_2^i$  in  $\mathcal{C}^4$  is

$$V_2(\sigma_2^i) = \frac{l^2}{2H^2} S \sqrt{\xi - 1/4}. \quad (14)$$

The volume of the 10 boundary 2-simplices,  $\sigma_2^b$  in  $\mathcal{C}^4$  is

$$V_2(\sigma_2^b) = \frac{\sqrt{3}l^2}{4H^2}S. \quad (15)$$

The volumes of the internal and boundary 3-simplices of  $\mathcal{M}^4$  are, respectively

$$V_3(\sigma_3^i) = \frac{l^3}{12H^3}S^{3/2}\sqrt{3\xi - 1}, \quad (16)$$

$$V_3(\sigma_3^b) = \frac{\sqrt{2}l^3}{12H^3}S^{3/2}. \quad (17)$$

There is only one kind of interior 2-simplex and boundary 2-simplex. The dihedral angle associated with each interior 2-simplex is

$$\theta(\sigma_2^i) = \arccos \frac{2\xi - 1}{6\xi - 2}. \quad (18)$$

for the boundary 2-simplices we have

$$\theta(\sigma_2^b) = \arccos \frac{1}{2\sqrt{6\xi - 2}}. \quad (19)$$

With respect to the matter terms, the kinetic term vanishes when the edges  $\sigma_2$  are boundary edges. The only non-vanishing contribution comes from the internal edges  $\sigma_2 = [0j]$ .

Computing the relevant volumes associated with the internal edges and all the vertices we conclude that the Regge action for this simplicial minisuperspace is

$$\begin{aligned} I[\xi, S, \phi_i, \phi_b] &= -\frac{S}{H^2} \left\{ \left( 5\sqrt{3} - \frac{5}{2}\sqrt{3}\eta\phi_b^2l^2 \right) \left[ \pi - 2\arccos \frac{1}{2\sqrt{6\xi - 2}} \right] \right. \\ &+ \left[ 10 - \frac{5}{3}\eta(\phi_i^2l^2 + 2\phi_b^2l^2) \right] \sqrt{\xi - 1/4} \left[ 2\pi - 3\arccos \frac{2\xi - 1}{6\xi - 2} \right] \\ &- \left( \frac{1}{24\sqrt{2}} \right) \frac{\sqrt{\xi - 3/8}}{\xi} (\phi_i l - \phi_b l)^2 \left. \right\} + \frac{S^2}{H^2} \left\{ \left( \frac{5}{4\sqrt{2}} \right) \sqrt{\xi - 3/8} \right. \\ &+ \left. \frac{1}{48\sqrt{2}} \left( \frac{m^2 l^2}{H^2} \right) \sqrt{\xi - 3/8} (\phi_i^2 l^2 + 4\phi_b^2 l^2) \right\} \end{aligned}$$

Note that we will be expressing the values of the field  $\phi$  and its mass  $m$ , in Planck units ( $l^{-1}$ ).

Thus in order to approximate the formal sum over histories by a fully computable expression

$$\Psi[\alpha_4; s_b; \phi_b] = \int_C D\xi D\phi_i e^{-I[S, \xi; \phi_b, \phi_i]}, \quad (20)$$

we only need to specify the integration contour  $C$ , and the measure of integration  $D\xi D\phi_i$ .

As in the previous cases studied by us [6], [9], in this simplified model the result yielded by a certain contour  $C$  is not very sensitive to the choice of measure if we stick to the usual measures, i.e., polynomials of the squared edge lengths. In our case we take

$$D\xi D\phi_i = \frac{ds_i}{2\pi i l^2} d\phi_i = \frac{S}{2\pi i H^2} d\xi d\phi_i \quad (21)$$

Since in the case of closed cosmologies there is as yet no known explicit prescription for the integration contour, one usually takes a pragmatic view, in which we look for contours that lead to the desired features of the wavefunction of the universe. Following [11], these features are:

- It should yield a convergent path integral
- The resulting wavefunction should predict classical spacetime in the late universe, i.e, oscillating behaviour when the  $\Psi$  is well approximated by the semiclassical approximation.
- The resulting wavefunction should obey the diffeomorphism constraints, in particular the Wheeler-DeWitt equation.

It is well known that any integration contour over real metrics would yield a wavefunction that does not satisfy any of these basic requirements. On the other hand, an integration contour over complex metrics can, if wisely chosen, lead to a wavefunction that does satisfy them.

In the simplicial framework, complex metrics arise from complex-valued squared edge lengths, (2). The boundary squared edge length,  $S$ , has to be real and positive for obvious physical reasons. But the interior squared edge length,  $\xi$ , can be allowed to take complex values.

Given the above requirements, [7] proposes that we use a steepest descents integration contour on the space of complex valued interior edge-lengths passing through the classical solutions that should dominate the wavefunction in the late universe, Before we test his proposal it is essential that we do the analytic study of the action as a multivalued function of the complex variable  $\xi$ .

### 3.2 Analytic Study of the Action

The action is trivially analytic with respect to the variables  $\phi_i$ ,  $\phi_b$  and  $S$ . But its dependence on the complex-valued  $\xi$  is much more complicated. So we shall investigate the analytic properties of  $I$  as if it was a function of  $\xi$  only,  $I = I[\xi]$ , the other variables acting as parameters.

The function  $I[\xi]$  has singularities at  $\xi = 0$  and  $\xi = 1/3$ , and square root branch points at  $\xi = 1/4$ ,  $1/3$  and  $3/8$ . These branch points correspond respectively to the vanishing of the volume of the internal 2-simplices, 3-simplices and 4-simplices. Using

$$\arccos z = -i \log(z + \sqrt{z^2 - 1})$$

we see that  $\xi = 1/3$  is also a logarithmic branch point, near which the action behaves like :

$$I \sim i2 \left( 5\sqrt{3} - \frac{5}{2}\sqrt{3}\eta\phi_b^2 l^2 \right) \left( \frac{S}{H^2} \right) \log(3\xi - 1) \quad (22)$$

The multivaluedness of  $I[\xi]$  associated with these branch points forces us to implement branch cuts in order to obtain a continuous function. In general, for terms of the type  $\sqrt{z - z_0}$  we consider a branch cut  $(-\infty, z_0]$ . So the branch cuts associated with the terms  $\sqrt{\xi - 3/8}$ ,  $\sqrt{\xi - 1/4}$  and  $\sqrt{\xi - 1/3}$ , altogether lead to a branch cut  $(-\infty, 3/8]$ . On the other hand, terms of the type  $\arccos(z)$  have branch points at  $z = -1, +1, \infty$ , and usually the associated branch cuts are chosen as  $(-\infty, -1] \cup [1, +\infty)$ . These terms are also infinitely multivalued.

The corresponding cuts for the term  $\arccos \frac{2\xi-1}{6\xi-2}$  are  $(\frac{1}{3}, \frac{3}{8}] \cup [\frac{1}{4}, \frac{1}{3})$ . On the other hand, associated with the term  $\arccos \frac{1}{2\sqrt{6\xi-2}}$  we have one cut  $(\frac{1}{3}, \frac{3}{8}]$  associated with  $\arccos u(z)$ , and another  $(-\infty, \frac{1}{3}]$  associated with  $u(z) = \sqrt{6\xi-2}$ .

So when we consider all these branch cuts simultaneously, we see that one way to obtain a continuous action  $I$  as a function of  $\xi$ , is to consider a total branch cut  $(-\infty, \frac{3}{8}]$ . Note that this also takes care of the singularity at  $\xi = 0$ . Although the action then becomes a continuous function of  $\xi$  in the complex plane with a cut  $(-\infty, \frac{3}{8}]$ , it is still infinitely multivalued. As usual in similar cases, in order to remove this multivaluedness we redefine the domain where the action is defined, from the complex plane to the Riemann surface associated with  $I$ . The infinite multivaluedness of the action reflects itself in  $I$  having an infinite number of branches with different values. The Riemann surface is composed of an infinite number of identical sheets,  $\mathbb{C} - (-\infty, \frac{3}{8}]$ , one sheet for each branch of  $I$ .

We define the first sheet  $\mathbb{C}_1$  of  $I[\xi]$  as the sheet where the terms in  $\arccos(z)$  assume their principal values. So the action in the first sheet will be formally equal to the original expression. Note that with the first sheet defined in this way, for real  $\xi > 3/8$  the volumes and deficit angles are all real, leading to a real Euclidean action for  $\xi \in [3/8, +\infty)$  on the first sheet. On the other hand, when  $\xi$  is real and less than  $1/4$  in the first sheet, the volumes become pure imaginary and the Euclidean action becomes pure imaginary. For all other points of this first sheet the action is fully complex.

When the action is continued in  $\xi$  once around all finite branch points ( $\xi = 1/4, 1/3, 3/8$ ), we reach what shall be called the second sheet. It is easy to conclude that the action in this second sheet is just the negative of the action in the first sheet.

Since by (2) we see that the simplicial metric in each 4-simplex is real *iff*  $\xi$  is real, then the simplicial geometries built out of these 4-simplices will be real when  $\xi$  is real. Furthermore the corresponding eigenvalues of  $g_{ij}$  are  $\lambda = \{4(\xi - 3/8), 1/2, 1/2, 1/2\}$ , [4]. So we see that for real  $\xi > 3/8$  we have real Euclidean signature geometries, with real Euclidean action, and for real  $\xi < 1/4$ , we have real Lorentzian signature geometries with pure imaginary Euclidean action.

### 3.3 Asymptotic Behaviour of the Action

If we are to compute the integration of  $e^{-I}$  along an SD contour, one of the essential things we have to know is the behaviour of the integrand, i.e., of the action, at infinity with respect to the variable  $\xi$ . Only then can we be confident that the integral converges, with the classical solutions dominating the wavefunction for the late universe.

It is easy to see that as  $\xi \rightarrow \infty$  the action behaves like

$$I[\xi, S, \phi_i, \phi_b] \sim \frac{\frac{5}{4\sqrt{2}} + \frac{1}{48\sqrt{2}}\left(\frac{m}{H}\right)^2\left(\phi_i^2 + 4\phi_b^2\right)}{H^2} S(S - S_{crit})\sqrt{\xi} \quad (23)$$

where

$$S_{crit} = \frac{\left[10 - \frac{5}{3}\eta\left(\phi_i^2 + 2\phi_b^2\right)\right][2\pi - 3\arccos(1/3)]}{\frac{5}{4\sqrt{2}} + \frac{1}{48\sqrt{2}}\left(\frac{m}{H}\right)^2\left(\phi_i^2 + 4\phi_b^2\right)} \quad (24)$$

The asymptotic behaviour of  $I$  for large  $\xi$  depends on whether or not  $S$  is larger than the critical value  $S_{crit}$ . However, contrary to the corresponding model in [6] where there was no scalar curvature coupling and the critical value of  $S$  was

$$S_{crit}^{\eta=0} = \frac{10[2\pi - 3 \arccos(1/3)]}{\frac{5}{4\sqrt{2}} + \frac{1}{48\sqrt{2}} \left(\frac{m}{H}\right)^2 (\phi_i^2 + 4\phi_b^2)} \quad (25)$$

and as such could only take values in a limited range, now  $S_{crit}$  can be arbitrarily negative or positive because of its  $\eta$  dependence. This means that the convergence of the integral along the SD contour, for a given  $S$ , will depend on the value of the coupling constant  $\eta$ .

## 4 Classical Solutions

The classical simplicial geometries are the extrema of the Regge action we have obtained above. In our minisuperspace model there are two degrees of freedom  $\xi, \phi_i$ , so the Regge equations of motion will be:

$$\frac{\partial I}{\partial \xi} = 0 \quad (26)$$

and

$$\frac{\partial I}{\partial \phi_i} = 0. \quad (27)$$

They are to be solved for the values of  $\xi, \phi_i$ , subject to the fixed boundary data  $S, \phi_b$ . The classical solutions will thus be of the form  $\bar{\xi}(S, \phi_b)$ , and  $\bar{\phi}_i(S, \phi_b)$ . The solution  $\bar{\xi}(S, \phi_b)$  completely determines the simplicial geometry.

Note that we shall be working on the first sheet. Of course, since on the second sheet the action is just the negative of this, the equations of motion are the same. And obviously every classical solution  $\bar{\xi}_I(S, \phi_b)$  located on the first sheet will have a counterpart  $\bar{\xi}_{II}$  of the same numerical value, but located on the second sheet, and so with an action of opposite sign,  $I[\bar{\xi}_I(S, \phi_b)] = -I[\bar{\xi}_{II}(S, \phi_b)]$ . So the classical solutions occur in pairs.

The classical equation (27) is

$$\phi_i = \frac{\phi_b}{A(\xi) + \frac{1}{2} \left(\frac{m^2}{H^2}\right) \xi S} \quad (28)$$

where

$$A(\xi) = 1 + 60\eta\xi\sqrt{2}\sqrt{\frac{\xi - 1/4}{\xi - 3/8}} \left[ 2\pi - 3 \arccos \frac{2\xi - 1}{6\xi - 2} \right] \quad (29)$$



Introducing this equation into the first one (26), we obtain a very long cubic equation in  $S$  for each value of  $\xi$ , given fixed  $\eta$  and  $\phi_b$ .

$$A_3(\xi)S^3 + A_2(\xi)S^2 + A_1(\xi)S + A_0(\xi) = 0, \quad (30)$$

where

$$A_3(\xi) = 30\left(K^2 + \frac{2}{15}K^3\phi_b^2\right)\xi^2, \quad (31)$$

$$\begin{aligned} A_2(\xi) = & -240\sqrt{2}\left(1 - \frac{\eta\phi_b^2}{3}\right)\sqrt{\frac{\xi - 3/8}{\xi - 1/4}}\left[2\pi - 3\arccos\frac{2\xi - 1}{6\xi - 2}\right]K^2\xi^2 + \\ & + 60\left(K + \frac{2}{15}K^2\phi_b^2\right)A(\xi)\xi - 20\eta K^2\phi_b^2\frac{\xi^2}{\xi - 1/3} - \left(\xi - \frac{3}{4}\right)K^2\phi_b^2 \end{aligned}$$

$$\begin{aligned} A_1(\xi) = & 30A(\xi)^2\left(1 + \frac{1}{6}K\phi_b^2\right) - 40\eta KA(\xi)\phi_b^2\frac{\xi}{\xi - 1/3} - 2K\phi_b^2[A(\xi) - 1]\frac{(\xi - 3/4)}{\xi} \\ & - 480\sqrt{2}\left(1 - \frac{\eta\phi_b^2}{3}\right)\sqrt{\frac{\xi - 3/8}{\xi - 1/4}}KA(\xi)\xi\left[2\pi - 3\arccos\frac{2\xi - 1}{6\xi - 2}\right] + [1 - A(\xi)^2]K\phi_b^2 \end{aligned}$$

$$\begin{aligned} A_0(\xi) = & -240\sqrt{2}\sqrt{\frac{\xi - 3/8}{\xi - 1/4}}\left(A(\xi)^2 - \frac{\eta\phi_b^2 A(\xi)^2}{3} - \frac{\eta\phi_b^2}{6}\right)\left[2\pi - 3\arccos\frac{2\xi - 1}{6\xi - 2}\right] \\ & - \frac{20\eta\phi_b^2[A(\xi)^2 - 1]}{\xi - 1/3} - [A(\xi) - 1]^2\phi_b^2\frac{(\xi - 3/4)}{\xi^2} \end{aligned}$$

with  $K = 1/2(m/H)^2$ .

This equation can then be solved numerically for  $S$ , and by inverting the resulting solutions we obtain several branches of solutions  $\xi = \xi_{cl}(S, \phi_b)$ . For obvious physical reasons we shall accept only solutions with real positive  $S$ . In figure 2 we show such solutions for  $\eta = 0.015$ ,  $m = 1$ , and  $\phi_b = 1$ . Between the critical points  $\xi = 1/4$  and  $\xi = 3/8$  there are no real solutions. We chose a small value of  $\eta$  and included negative branches so that the behaviour of the solutions near the critical points is clear. The behaviour of the corresponding solutions for larger values of  $\eta$  is of the same type, but the separation between the several branches is not so obvious.

It is easy to see that for any positive value of  $S$  there is at least one classical Lorentzian solution ( $\bar{\xi} < 1/4$ ). Furthermore, when  $S$  is large (late Universe) there is only one Lorentzian

solution,  $\bar{\xi}_I^L(S, \phi_b)$ , (in the first sheet, of course), and is located near the critical point  $\xi = 1/4$ , just like the results obtained in [6]. There is also, of course, its counterpart in the second sheet  $\bar{\xi}_{II}^L(S, \phi_b)$ , which though numerically equal has symmetrical action. However, the asymptotic behaviour of the classical solutions is very different from that in [6]. Unlike in [6], the Euclidean branch ( $\xi > 3/8$ ), is not limited to a finite range of  $S$ . In the present case the Euclidean branch goes all the way to  $+\infty$ , although this is not at all evident from figure 2 because it is a large-scale behaviour, and this is a small-scale picture. We can see this large-scale behaviour in figure 3, but at the cost of the Lorentzian peak at  $\xi = 1/4$  becoming indistinct from the imaginary  $\xi$  axis. In figure 3 we see that for each positive value of  $S$  there is always one pair of Euclidean signature solutions  $\bar{\xi}_I^E(S, \phi_b) = \bar{\xi}_{II}^E(S, \phi_b) \in [3/8, +\infty)$ , and  $\bar{\xi}^E \rightarrow +\infty$  as  $S \rightarrow +\infty$ .

If we look in the opposite direction, i.e.  $\xi \rightarrow -\infty$  another surprise awaits us. The positive Lorentzian branch eventually becomes negative and connects with the upper negative branch.

So the main difference introduced by the scalar coupling is the existence of Euclidean  $\bar{\xi}_I^E(S, \phi_b) = \bar{\xi}_{II}^E(S, \phi_b) \in [3/8, +\infty)$  solutions for all positive values of the boundary edge length  $S$ . In a semiclassical analysis this seems to mean that the Lorentzian universe can nucleate with arbitrary size from an Euclidean regime. However the Regge action grows very fast as  $\xi$  goes from  $3/8$  to  $+\infty$ , as we can see in figure 4. Consequently, the Euclidean classical solutions for large  $S$  are very strongly suppressed.

For negative values of  $\eta$  the solutions obtained have a very different behaviour, but they agree with the positive  $\eta$  solutions in some very important points. In order to present these solutions more clearly we have separated the Lorentzian range ( $\xi < 3/8$ ), presented in figure 5, from the Euclidean range ( $\xi > 3/8$ ), in figure 6. As can be seen in figure 5, where  $\eta = -0.15$ , despite the somewhat bizarre behaviour, we still have that for large values of  $S$  there is an unique pair of classical Lorentzian solutions  $\bar{\xi}_I^L(S, \phi_b) = \bar{\xi}_{II}^L(S, \phi_b)$  and these will be located near the critical point  $\xi = 1/4$ . Also, in the Euclidean regime,  $]3/8, +\infty)$ , there is an unique pair of Euclidean solutions  $\bar{\xi}_I^E(S, \phi_b) = \bar{\xi}_{II}^E(S, \phi_b) \in [3/8, +\infty)$  for any positive value of  $S$ , and those solutions go to  $+\infty$  as  $S \rightarrow +\infty$ , as shown in figure 6.

## 5 Steepest Descents Contour

After studying the analytical and asymptotic properties of the action we can now focus on the Euclidean path integral that yields the wave function of the Universe (20), (21).

As we have mentioned above there is as yet no universally accepted prescription for the integration contour  $C$  to use in quantum cosmology. Following Hartle [7], we shall accept that the main criteria any contour should satisfy are that it should lead to a convergent path integral and to a wave function predicting classical Lorentzian spacetime in the late Universe. The steepest descents contour over complex metrics seems to be the leading candidate. In the simplicial framework, complex metrics arise from complex-valued squared edge lengths, (2).

We shall look for the steepest descent (SD) contour, thinking of the action as a function of only the complex variable  $\xi$ , and for the moment consider  $\phi_i$  to be only a real parameter in  $I = I[\xi]$  to be integrated over later. In general, an SD contour associated with an extremum ends up either at  $\infty$ , at a singular point of the integrand, or at another extremum with the same value of  $Im(I)$ . We have seen that when  $S$  is big enough the only classical solutions are a pair of real Lorentzian solutions ( $\bar{\xi}_I^L(S, \phi_b) = \bar{\xi}_{II}^L(S, \phi_b) \rightarrow 1/4$ ), and a pair of Euclidean solutions ( $\bar{\xi}_I^L(S, \phi_b) = \bar{\xi}_{II}^L(S, \phi_b) \gg 3/8$ ).

In both cases for each pair of solutions one of these solutions is located on the first sheet and the other on the second sheet. Thus, they have pure imaginary actions of opposite sign in the Lorentzian case and symmetric real actions in the Euclidean case. Given that their actions are different valued, in each pair there is no single SD path can go directly from one solution the other extremum, but it is possible for the two sections to meet in infinity, and together they form the total SD contour. In effect given that

$$I[\bar{\xi}] = [I[\bar{\xi}^*]]^*$$

and

$$I[\bar{\xi}_I] = -I[\bar{\xi}_{II}]$$

where  $*$  denotes complex conjugation, we see that the SD path that passes through  $\bar{\xi}_{II}$  will be the complex conjugate of the SD path that passes through  $\bar{\xi}_I$ . So the *total* SD contour will always be composed of two complex conjugate sections, each passing through one extremum, and this together with the real analyticity of the action guarantees that the resulting wavefunction is real.

For large  $S$  we choose to consider the SD contour associated with the Lorentzian solutions for two reasons. First, it is the only one likely to describe a late universe like our own. Second, the Euclidean action of the Euclidean solutions becomes very large very fast when  $S$

increases, which strongly suppresses these solutions in any computation of the wavefunction, except when  $S$  is small, (see figure 4).

The SD path in the complex  $\xi$  Riemann surface of  $I$ , passing through a generic classical solution  $\{\xi_{cl}(S, \phi_b), \phi_i^{cl}(S, \phi_b)\}$  is defined as:

$$C_{SD}(S, \phi_b, \phi_i) = \left\{ (\xi \in R) : Im[I(S, \xi, \phi_i, \phi_b)] = \tilde{I}[\xi_{cl}(S, \phi_b), \phi_i^{cl}(S, \phi_b)] \right\} \quad (32)$$

where  $R$  is the Riemann sheet of the action, and  $\tilde{I}(\xi) = iI(\xi)$ .

In figure 7 we show the result of a numerical computation of this path for  $m = 1$ ,  $\phi_b = 1$ ,  $\eta = 0.015$ ,  $S = 150$ , and  $\phi_i = 0.15$ .

The SD path associated with the the other solution in the second sheet is just the mirror image of this, relative to the real  $\xi$  axis. Together they form the SD contour we are looking for.

Changes in the values of  $\phi_b, \phi_i$  and  $\eta$ , do not alter the generic shape of this SD contour. It starts at  $+\infty$  in the first quadrant and ends in the real  $\xi$  axis precisely at the classical solution to which it is associated.

Going upward from the Lorentzian extremum on the first sheet, the SD contour proceeds to infinity in the first quadrant approximately along the parabola

$$\begin{aligned} & \left[ \frac{5}{4\sqrt{2}} + \frac{1}{48\sqrt{2}} \left( \frac{m}{H} \right)^2 (\phi_i^2 + 4\phi_b^2) \right] \\ & \times \frac{S}{H^2} (S - S_{crit}) Im(\sqrt{\xi}) = \tilde{I}[\xi_{cl}^L, \phi_{cl}] \end{aligned}$$

The convergence of the integral along this part of the contour is dependent on the the asymptotic behaviour of the real part of the Euclidean action on the first quadrant of the first sheet

$$\begin{aligned} Re[I^I(\xi, S, \phi_i, \phi_b)] & \sim \left[ \frac{5}{4\sqrt{2}} + \frac{1}{48\sqrt{2}} \left( \frac{m}{H} \right)^2 (\phi_i^2 + 4\phi_b^2) \right] \\ & \times \frac{S}{H^2} (S - S_{crit}^I) \sqrt{|\xi|} \end{aligned}$$

When there was no scalar curvature coupling, as in [6], there was a finite maximum value that  $S_{crit}$  could take, no matter what the values of the parameters  $m$  and  $\phi_b$  were. This guaranteed convergence of the integral in the first quadrant, from a certain value of  $S$ ,

whatever the values of  $m$  and  $\phi_b$ . Now the situation is different because  $S_{crit}$ , (24), includes a term in  $\eta$  that makes  $S_{crit} \rightarrow +\infty$  as  $\eta \rightarrow -\infty$ .

## 6 Semiclassical Approximation

One of the main requirements on any model is that it yields a wavefunction that in the late universe predicts a classical (Lorentzian) spacetime, like the one we experience. Now, a wavefunction of the universe will predict a classical spacetime where it is well approximated by the semiclassical approximation associated with Lorentzian classical solutions. From what we have seen above, the SD contour passing through the classical Lorentzian solutions satisfies that condition for large enough  $S$ .

In our model there are two integration variables  $\xi$  and  $\phi_i$  and the full wavefunction of the universe is given by (10). We work under the assumption that  $\phi_i$  is to be integrated over real values, and  $\xi$  over the complex Riemann surface of the Euclidean action  $I$ . We now know that the integral over  $\xi$  can be calculated as a steepest descent ( $SD$ ) integral for all the relevant values of  $\phi_i$ , and that the action peaks about the classical solutions  $\bar{\xi}$ .

We can thus replace  $\int_{C_{SD}} d\xi e^{-I}$  by its semiclassical approximation based on the relevant classical extrema. This will give rise to Laplace type integrals in  $\phi_i$ , when the extrema have real Euclidean action, and to Fourier type integrals in  $\phi_i$ , when the extrema have pure imaginary Euclidean action. These integrals can then be shown to be dominated by the stationary points of the integrand which coincide with the classical solutions  $\bar{\phi}_i$ , where

$$\left. \frac{\partial I[\bar{\xi}, \phi_i]}{\partial \phi_i} \right|_{\phi_i = \bar{\phi}_i} = 0$$

This justifies the validity of the semiclassical approximation to the full wavefunction.

So given the full wavefunction

$$\Psi(S, \phi_b) = \frac{S}{2\pi i H^2} \int_C d\xi d\phi_i e^{-I(\xi, S, \phi_i, \phi_b)}, \quad (33)$$

with an SD contour associated with real classical Lorentzian solutions  $\{\bar{\xi}_k(S, \phi_b)\}$  with pure imaginary actions  $I_k = i\tilde{I}[\bar{\xi}_k(S, \phi_b); \phi_i] = i\tilde{I}_k(S, \phi_b, \phi_i)$ , the semiclassical approximation of the wavefunction will be

$$\Psi_{SC}(S, \phi_b) \sim \int d\phi_i \sum_k \sqrt{\frac{S^2}{2\pi H^4 |\tilde{I}''[\bar{\xi}_k(S, \phi_b), \phi_i]|}} e^{-i[\tilde{I}(\bar{\xi}_k(S, \phi_b), \phi_i) + \mu_k \frac{\pi}{4}]}$$

$$\sim \sum_k \sqrt{\frac{S^2}{2\pi H^4 |\tilde{I}_k''(S, \phi_b)|}} e^{-i[\tilde{I}_k(S, \phi_b) + \mu_k \frac{\pi}{4}]}$$

where ' means derivative with respect to  $\xi$ , and  $\mu_k = \text{sgn}(\tilde{I}'')$ .

When the dominant extrema are real Euclidean solutions  $\{\bar{\xi}_k(S, \phi_b)\}$ , with real Euclidean actions, then after the semiclassical evaluation of the integral over  $\xi$ , we are left with Laplace-type integrals over  $\phi_i$  which are dominated by the contributions coming from the stationary points of  $I[\bar{\xi}_k(S, \phi_b), \phi_i]$ , which are precisely the classical solutions  $\bar{\phi}_i^k(S, \phi_b)$ . So the semiclassical wavefunction will then be

$$\Psi_{SC}(S, \phi_b) \sim \sum_k \sqrt{\frac{S^2}{2\pi H^4 |I_k''(S, \phi_b)|}} e^{-I_k(S, \phi_b)} \quad (34)$$

Since we are mainly interested in knowing if this model predicts classical Lorentzian spacetime for the late universe, we have computed the semiclassical wavefunction associated with the classical Lorentzian branch of solutions near  $\xi = 1/4$  in figure 2. We considered  $\eta = 0.225, m = 1$  and  $\phi_b = 1$ , and the result obtained is shown in figure 8. The behaviour exhibited during the late universe, (i.e. large values of  $S$ ) is typical of that of a wavefunction describing a classical Lorentzian universe, as desired.

As for the Euclidean solutions, their semiclassical contribution is exponentially suppressed except for small values of  $S$ ; see figure 9. Furthermore, it should be noted that this suppression becomes increasingly strong as  $\eta$  grows. The peak in the semiclassical wavefunction is not caused by the behaviour of the Euclidean action, which is monotonically increasing as  $S$  increases, but by the pre-factor involving the second derivative of the action.

It is clear that although there are classical Euclidean solutions for any value of  $S$  the probability associated with a Euclidean universe with large boundary, (large  $S$ ), is very small, and so the late universe wavefunction should be well approximated by the semiclassical wavefunction associated with the Lorentzian solutions.

## 7 Conclusions

The addition of the arbitrary scalar coupling  $\eta R\phi^2$  has produced results never seen in all of the previous simplicial models considered in [7], [4], [6] and [9]. First, we note the existence of real Euclidean classical solutions for any size of the boundary *three*-space. However, the contribution of these solutions to the wavefunction of the universe was seen to be exponentially suppressed except for small boundary *three*-spaces. In the late universe the

wavefunction was found to be dominated by the contribution of classical Lorentzian solutions. Second, the SD contours passing through these solutions are different from the ones obtained in [6]. The SD path associated to the Lorentzian solution in the first sheet of Riemann surface of  $I[\xi]$ , starts off at infinity moving downwards through the first quadrant, ending up precisely at the classical Lorentzian solution, and does not cross into the second sheet as before. Third, the behaviour of the real part of the action in the first quadrant now depends on the value of  $\eta$  relative to the value of  $S$ , because  $S_{crit} \rightarrow +\infty$  as  $\eta \rightarrow -\infty$ . So we see that since the value of  $\eta$  is arbitrary so is the value  $S_{crit}$  from which the SD path converges.

Nevertheless for any given  $\eta$  there is a value of  $S$  from which the SD integral is convergent and is dominated by the contribution of the Lorentzian classical solutions. Furthermore, larger values of  $S$  lead to a stronger peak in the action around those classical solutions and this makes the semiclassical approximations of the SD wavefunction quite good. The oscillatory behaviour of the semiclassical wavefunctions indicates that the wavefunction of the universe for this model predicts classical Lorentzian spacetime for the late universe, (large  $S$ ).

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